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On k-connectivity problems with sharpened triangle inequality ☆

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ABSTRACT

The k-connectivity problem is to find a minimum-cost k-edge- or k-vertex-connected spanning subgraph of an edge-weighted, undirected graph G for any given G and k. Here, we consider its NP-hard subproblems with respect to the parameter β, with $\frac{1}{2} < \beta < 1$, where $G = (V, E)$ is a complete graph with a cost function c satisfying the sharpened triangle inequality $c(\{u,v\}) \leqslant \beta \cdot (c(\{u,w\}) + c(\{w,v\}))$ for all $u, v, w \in V$.
First, we give a simple linear-time approximation algorithm for these optimization problems with approximation ratio $\frac{\beta}{1-\beta}$ for any $\frac{1}{2} \leqslant \beta < 1$, which improves the known approximation ratios for $\frac{1}{2} < \beta < \frac{2}{3}$.
The analysis of the algorithm above is based on a rough combinatorial argumentation. As the main result of this paper, for $k = 3$, we sophisticate the combinatorial consideration in order to design a $(1 + \frac{5(2\beta-1)}{9(1-\beta)} + \mathcal{O}(\frac{1}{|V|}))$-approximation algorithm for the 3-connectivity problem on graphs satisfying the sharpened triangle inequality for $\frac{1}{2} \leqslant \beta \leqslant \frac{2}{3}$.
As part of the proof, we show that for each spanning 3-edge-connected subgraph H, there exists a spanning 3-regular 2-vertex-connected subgraph H' of at most the same cost, and H can be transformed into H' efficiently.

1. Introduction

In order to attack hard optimization problems that do not admit any polynomial-time approximation scheme (PTAS) or α-approximation algorithm for a reasonable constant α (or with an even worse approximability) one can consider the concept of *stability of approximation* [8,10,22,23]. The idea behind this concept is to find a parameter (characteristic) of the input instances that captures the hardness of particular inputs. An approximation algorithm is called *stable* with respect to this parameter, if its approximation ratio grows with this parameter but not with the size of the input instances. This approach is similar to the concept of parameterized complexity introduced by Downey and Fellows [17,18]. (The difference is in that we relate the parameter to the approximation ratio while Downey and Fellows relate the parameter to the time complexity.)

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A nice example is the Traveling Salesman Problem (TSP) that does not admit any polynomial-time approximation algorithm with an approximation ratio bounded by a polynomial in the size of the input instance, but is $\frac{3}{2}$-approximable for metric input instances. Here, one can characterize the input instances by their "distance" to metric instances. This can be expressed by the so-called β-triangle inequality for any $\beta \geqslant \frac{1}{2}$. For any complete graph $G = (V, E)$ with a cost function $c : E \to \mathbb{Q}_{>0}$ we say that (G, c) satisfies the β-triangle inequality, if

$$c\big(\{u, v\}\big) \leqslant \beta \cdot \big(c\big(\{u, w\}\big) + c\big(\{w, v\}\big)\big)$$

for all vertices $u, v, w \in V$. In the case of $\beta < 1$ we speak about the *sharpened triangle inequality*, and if $\beta > 1$ we speak about the *relaxed triangle inequality*. Note that in the case of $\beta = 1$ we have the well-known metric TSP, and if $\beta = \frac{1}{2}$, the problem becomes trivial since all edges must have the same cost. For a detailed motivation of the study of TSP instances satisfying sharpened triangle inequalities, see [7].

In a sequence of papers [1,2,4,7–9,11] it was shown that

1. there are stable approximation algorithms for the TSP whose approximation ratio grows with β, but is independent of the size of the input, and
2. for every $\beta > \frac{1}{2}$ one can prove explicit lower bounds on the polynomial-time approximability growing unboundedly with β.

Thus, one can partition the set of all input instances of the TSP into infinitely many classes with respect to their hardness, and one gains the knowledge that hard TSP input instances have to have small edge costs as well as edge costs of exponential size in the size of G.

A natural question is whether there are other problems for which the triangle inequality can serve as a measure of hardness of the input instances. In [5] it is shown that this is the case for the problem of constructing 2-connected spanning subgraphs of a given complete graph whose edge weights obey the sharpened triangle inequality.

Here, we consider a more general problem: For a given positive integer $k \geqslant 2$ and an edge-weighted graph G, one has to find a minimum k-edge- or k-vertex-connected spanning subgraph. This problem is well-known to be NP-hard [20]. Concerning approximability results, the k-edge-connected subgraph problem is approximable within 2 [26].[1] In case of the k-vertex-connected subgraph problem, the best known approximation ratio in general is $\mathcal{O}(\ln k)$ for $|V| \geqslant 6k^2$ [14], and $\mathcal{O}(\ln k \min\{\sqrt{k}, \frac{n}{n-k} \ln k\})$ otherwise [28]. In case $k = 3$, a 2-approximation algorithm exists [3], and in case $k = 4, 5$, a 3-approximation algorithm is known [16]. Comprehensive surveys on the minimum k-edge- and k-vertex-connected spanning subgraph problem can be found in [24,29].

As far as the metric case is concerned, the best known ratio is $2 + \frac{k-1}{|V|}$ [27]. Furthermore, it is easy to see that for the β-sharpened triangle inequality, such an algorithm has ratio $2 + \frac{k\beta}{|V|}$. In this paper, we first easily improve this bound for each $\frac{1}{2} \leqslant \beta \leqslant \frac{2}{3}$, by providing a $\frac{\beta}{1-\beta}$-approximation algorithm. This result is based on the simple observation that the costs of two edges adjacent to the same vertex may not differ too much, if the input satisfies the sharpened triangle inequality. Some rough combinatorial calculations show that the cost of an optimal k-edge-connected subgraph does not differ too much from the cost of any k-regular subgraph.

The main contribution of this paper is then to enhance this approach to a more sophisticated technique providing:

1. A polynomial-time $(1 + \frac{5}{9} \cdot \frac{2\beta-1}{1-\beta})$-approximation algorithm for the 3-edge- and 3-vertex-connectivity problem on graphs with an even number of vertices satisfying the sharpened triangle inequality for any $\frac{1}{2} \leqslant \beta \leqslant \frac{2}{3}$. Note that this approximation ratio tends to 1 for $\beta \to \frac{1}{2}$, and it is $\frac{14}{9}$ for $\beta = \frac{2}{3}$.
2. A polynomial-time $(1 + \frac{5}{9} \cdot \frac{2\beta-1}{1-\beta} + \mathcal{O}(\frac{1}{|V|}))$-approximation algorithm for the 3-edge- and 3-vertex-connectivity problem on graphs with an odd number of vertices satisfying the sharpened triangle inequality for any $\frac{1}{2} \leqslant \beta \leqslant \frac{2}{3}$. Note that this approximation ratio tends to $1 + \mathcal{O}(\frac{1}{|V|})$ for $\beta \to \frac{1}{2}$, and it is $\frac{14}{9} + \mathcal{O}(\frac{1}{|V|})$ for $\beta = \frac{2}{3}$.

As part of the proof of the approximation ratio, we obtain the following result that is interesting in itself.

3. For each spanning 3-edge-connected subgraph H, there exists a spanning 3-regular 2-vertex-connected subgraph H' of at most the same cost, and H can be transformed into H' efficiently.

Note that we start with a graph where there exist only edge-disjoint paths and we end with one having vertex-disjoint paths.

The paper is organized as follows. In Section 2 we will formally define the k-connectivity problem and provide some useful facts about graphs satisfying a sharpened triangle inequality. Section 3 is devoted to a linear-time approximation

[1] Some better results can be obtained for some restricted cases like unweighted graphs [13,19,25], and Euclidean input instances [12].

algorithm for the k-connectivity problem, and in Section 4 we will present our main result, namely an improved approximation algorithm for the 3-connectivity problem. Section 5 presents the proof of Claim 3 from above. We note that the proof techniques used in this paper essentially differ from the approaches used in the previous papers devoted to the approximability of input instances satisfying sharpened or relaxed triangle inequalities.

A short version of this paper has appeared in [6].

2. Preliminaries

Next we formulate the tasks which will be investigated in the rest of the paper. Recall that a graph is said to be *k-edge-connected* (k-ec, for short) if the removal of any $k-1$ edges leaves the graph connected. Similarly, a graph is said to be *k-vertex-connected* (k-vc, for short) if the removal of any $k-1$ vertices leaves the graph connected. Notice that every k-vc graph is also a k-ec graph. (Note that all graphs throughout this paper are considered to be undirected and simple.)

Definition 1. Let (G, c) be a weighted complete graph, where $G = (V, E)$, $c : E \to \mathbb{Q}_{>0}$, and let k be a positive integer. The *k-ec spanning subgraph* (k-*ECSS*) *problem* is that of computing a minimum-weight spanning k-ec subgraph $G' = (V, E')$ of G.

Definition 2. Let (G, c) be a weighted complete graph, where $G = (V, E)$, $c : E \to \mathbb{Q}_{>0}$, and let k be a positive integer. The *k-vc spanning subgraph* (k-*VCSS*) *problem* is that of computing a minimum-weight spanning k-vc subgraph $G' = (V, E')$ of G.

In this paper, we will focus on the k-ECSS/k-VCSS and the 3-ECSS/3-VCSS problems for graphs obeying the sharpened triangle inequality as defined in the introduction, and therefore we give some basic properties of weighted graphs satisfying the sharpened triangle inequality.

Lemma 1. *(See [7].) Let (G, c) be a weighted complete graph with $G = (V, E)$, $c : E \to \mathbb{Q}_{>0}$, where c obeys the sharpened triangle inequality for $\frac{1}{2} \leqslant \beta < 1$. Let $c_{\min}$ and $c_{\max}$ denote the minimum and the maximum edge cost occurring in G, respectively. Then*:

(a) *For all adjacent edges $e_1, e_2 \in E$, the inequality $c(e_1) \leqslant \frac{\beta}{1-\beta} c(e_2)$ holds.*

(b) $c_{\max} \leqslant \frac{2\beta^2}{1-\beta} c_{\min}$.

Corollary 1. *(See [5].) Let (G, c) be a weighted complete graph with $G = (V, E)$, $c : E \to \mathbb{Q}_{>0}$, where c obeys the sharpened triangle inequality for $\frac{1}{2} \leqslant \beta \leqslant \frac{2}{3}$. Then $c(e_3) \leqslant c(e_1) + c(e_2)$ for all edges $e_1, e_2, e_3 \in E$, where both e_1 and e_2 are adjacent to e_3.*

This result implies that in the case where $\beta \leqslant \frac{2}{3}$ holds, we can replace two edges by one adjacent edge without increasing the cost.

It has been shown in [5] that the 2-ECSS/2-VCSS problem for graphs obeying the sharpened triangle inequality for $\beta < \frac{2}{3}$ corresponds to the problem of finding a minimum cost Hamiltonian cycle, i.e., to the TSP in that graph. In [7] a $(\frac{2}{3} + \frac{1}{3} \cdot \frac{\beta}{1-\beta})$-approximation algorithm for the TSP for graphs obeying the sharpened triangle inequality, i.e. for $\frac{1}{2} \leqslant \beta < 1$, has been proposed, which we will briefly recall here, since we will apply it in Section 4 of this paper.

Algorithm Cycle Cover

Input: A weighted complete graph (G, c), where $c : E \to \mathbb{Q}_{>0}$ obeys the sharpened triangle inequality.
Step 1: Construct a minimum cost cycle cover $C = \{C_1, \ldots, C_k\}$ of G.
Step 2: For $1 \leqslant i \leqslant k$, find the cheapest edge $\{a_i, b_i\}$ in every cycle C_i of C.
Step 3: Obtain a Hamiltonian cycle H of G from C by replacing the edges $\{\{a_i, b_i\} \mid 1 \leqslant i \leqslant k\}$ by the edges $\{\{b_i, a_{i+1}\} \mid 1 \leqslant i \leqslant k-1\} \cup \{\{b_k, a_1\}\}$.
Output: H.

Furthermore, let us recall that a graph is called *k-regular*, iff each of its vertices has degree exactly k. Additionally, note that for any graph G that is k-regular, it must hold that $k \cdot n$ is even, where n is the number of vertices in G. This is due to the following argument. If $G = (V, E)$ is k-regular then the number of edges in G is $|E| = \frac{k \cdot n}{2}$. Thus either n or k has to be even.

3. An approximation algorithm for the k-ECSS/k-VCSS problem

In this section we will investigate an algorithm for the k-ECSS/k-VCSS problem for graphs obeying the sharpened triangle inequality for $\frac{1}{2} \leqslant \beta < 1$. We will proceed as follows.

Let (G, c) be a weighted complete graph such that c satisfies the sharpened triangle inequality. Let n be the number of the vertices, and let, for now, $k \cdot n$ be even. We will deal later with the case of odd $k \cdot n$, where we need a small additional adaption.

(i) We will prove that the cost of an arbitrary k-regular graph (G', c) differs from the cost of any k-edge-connected graph (G'', c) at most by a factor $\frac{\beta}{1-\beta}$, where G' and G'' are both spanning subgraphs of the same complete weighted graph (G, c).
(ii) We will describe a strategy to construct a spanning subgraph of (G, c) that is both k-vertex-connected and k-regular.

Thus, the algorithm proposed in (ii) will compute a k-vertex-connected spanning subgraph that $\frac{\beta}{1-\beta}$-approximates a minimum k-edge-connected spanning subgraph. Here we see why both problems can be treated in common. Each k-vertex-connected subgraph is also k-edge-connected which implies that a minimal k-vertex-connected spanning subgraph is at least as expensive as a k-edge-connected one.

Theorem 1. *Let (G, c), $G = (V, E)$, $c : E \to \mathbb{Q}_{>0}$ be a weighted complete graph obeying the sharpened β-triangle inequality ($\frac{1}{2} \leqslant \beta < 1$), and let $k \cdot |V|$ be even. Let $G' = (V, E')$ be an arbitrary k-regular spanning subgraph of G, and let $G'' = (V, E'')$ be an arbitrary k-edge-connected spanning subgraph of G. Then* $\mathrm{cost}(G') \leqslant \frac{\beta}{1-\beta} \cdot \mathrm{cost}(G'')$.

Proof. The idea is to pairwise compare adjacent edges in G' and G'', since, if an edge $e' \in E'$ is adjacent to an edge $e'' \in E''$, according to Lemma 1, their costs cannot differ by more than a factor $\frac{\beta}{1-\beta}$.

Hence, for each vertex v in V, denote by I'_v all edges in E' that are incident with v and by I''_v all edges in E'' incident with v. Note that the number of edges in I'_v is exactly k due to the k-regularity of G', and the number of edges in I''_v is at least k due to the k-edge-connectivity of G''. Since the edges in I'_v and I''_v are pairwise adjacent we can apply Lemma 1 and obtain

$$\mathrm{cost}(I'_v) \leqslant \frac{\beta}{1-\beta} \cdot \mathrm{cost}(I''_v). \tag{1}$$

Thus, we can estimate the costs of G' as follows:

$$\begin{aligned}
\mathrm{cost}(G') &= \sum_{e' \in E'} \mathrm{cost}(e') = \frac{1}{2} \sum_{v \in V} \sum_{e' \in I'_v} \mathrm{cost}(e') \\
&\leqslant \frac{1}{2} \sum_{v \in V} \sum_{e'' \in I''_v} \frac{\beta}{1-\beta} \mathrm{cost}(e'') \\
&= \frac{\beta}{1-\beta} \sum_{e'' \in E''} \mathrm{cost}(e'') = \frac{\beta}{1-\beta} \mathrm{cost}(G''). \qquad \square
\end{aligned}$$

Next, we present an algorithm that, for all integers k and n, where $n > k+1$ and $k \cdot n$ even, constructs a graph that has n vertices and is k-regular. Such a graph is known in the literature as the *Harary graph* [21], and retains the property of being the k-vertex-connected graph of n vertices with the smallest number of edges.

The idea is to start with a cycle of n vertices and to iteratively add all edges that connect vertices at distance i for $2 \leqslant i \leqslant \lfloor \frac{k}{2} \rfloor$. In the case of an odd value of k we additionally connect the vertices in a spoke-like way.

Algorithm *k*C-Graph

Input: Integers k and n, $k \geqslant 2$ and $n \geqslant k+1$, where $k \cdot n$ is even.
Step 1: Let $V = \{v_0, \ldots, v_{n-1}\}$.
Step 2: Let $E' = \{\{v_i, v_{(i+j) \bmod n}\} \mid 1 \leqslant j \leqslant \lfloor \frac{k}{2} \rfloor, 0 \leqslant i < n\}$
Step 3: If k is odd, then let $E = E' \cup \{\{v_i, v_{i+\frac{n}{2}}\} \mid 0 \leqslant i < \frac{n}{2}\}$,
else let $E = E'$.
Output: Graph $G = (V, E)$.

Thus, each vertex v_i in the graph G produced by the above algorithm is directly connected to all vertices that are within distance $\lfloor \frac{k}{2} \rfloor$ according to the basic cycle $v_0, v_1, \ldots, v_{n-1}, v_0$. For an example see Fig. 1.

We obtain the following result. Note that linear time means linear in the number of edges.

Theorem 2. *For all inputs (G, c) for the k-ECSS/k-VCSS problem, where c obeys the sharpened triangle inequality w.r.t. $\frac{1}{2} \leqslant \beta < 1$, there exists a linear-time $\frac{\beta}{1-\beta}$-approximation algorithm.*

Proof. If $k \cdot n$ is even, where n is the number of vertices in G, we can take the output of Algorithm kC-Graph and interpret it as a spanning subgraph of G. Then, the claim is a direct consequence from Theorem 1 and the correctness of the Algorithm kC-Graph, which runs in linear-time.

In the case where $k \cdot n$ is odd, one can consider an algorithm that is similar to Algorithm kC-Graph. This algorithm first finds an edge $e_{\min}$ of minimum cost in the graph, puts it into the solution, and proceeds by determining a graph by a similar

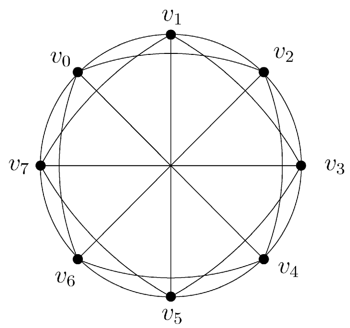

Fig. 1. The Harary graph for $n = 8$ and $k = 5$.

construction as in Algorithm kC-Graph in such a way that $e_{\min}$ is not considered there. The naming of the vertices is simply chosen such that $e_{\min} = \{v_{\lfloor \frac{n}{2} \rfloor}, v_{n-1}\}$ while the "spoke edges", the edges added in Step 3, are $\{\{v_i, v_{i+\lfloor \frac{n}{2} \rfloor}\} \mid 0 \leqslant i < \lfloor \frac{n}{2} \rfloor\}$.

That way, each vertex is connected to at least one other that is $\lfloor \frac{n}{2} \rfloor$ edges away along the perimeter, and $v_{\lfloor \frac{n}{2} \rfloor}$ is connected to two such vertices. Consequently, the proof of k-vertex-connectivity holds up as before.

Now we can estimate the costs of the resulting k-vertex-connected graph against the optimal solution in a similar way as in Theorem 1 by additionally taking into account that the optimal solution has at least $\lfloor \frac{k \cdot n}{2} \rfloor + 1$ edges while the constructed solution has exactly $\lfloor \frac{k \cdot n}{2} \rfloor + 1$ edges. Thus, the only edge which could not be estimated like in Theorem 1 is $e_{\min}$ having minimal cost, and hence can be effectively estimated against the additional edge occurring in the optimal solution. $\square$

The original idea behind the presented Algorithm kC-Graph (now hidden behind the combinatorial argument in the proof of Theorem 1) was based on the following two-step consideration.

1. For any optimal k-edge-connected subgraph G_{opt} of G satisfying the sharpened triangle inequality for $\frac{1}{2} \leqslant \beta < \frac{2}{3}$, one can construct a k-regular subgraph G_k (not necessarily k-edge-connected) with $\text{cost}(G_k) \leqslant \text{cost}(G_{\text{opt}})$.
2. For all k-regular spanning subgraphs G' and G'' of G,
$$\text{cost}(G') \leqslant \frac{\beta}{1-\beta} \text{cost}(G'').$$

Thus, any 3-regular 3-vertex-connected spanning subgraph is a good feasible solution for the 3-ECSS/3-VCSS problem. In the next section we improve this concept for $k = 3$ and $\beta \leqslant \frac{2}{3}$ in the sense that we apply the rough combinatorial argument (providing the approximation ratio $\frac{\beta}{1-\beta}$) only for a subpart of the graph G_k. More precisely, we exchange the steps 1 and 2 for

1′ For any optimal 3-edge-connected subgraph G_{opt} of G satisfying the sharpened triangle inequality for $\frac{1}{2} \leqslant \beta \leqslant \frac{2}{3}$, one can construct a 3-regular 2-vertex-connected subgraph G_3 with
$$\text{cost}(G_3) \leqslant \text{cost}(G_{\text{opt}}).$$

2′ G_3 can be partitioned into a 1-factor M and a 2-factor C. The cost of the 1-factor can be bounded from above in the same way as in 2, but the cost of C can be better approximated.

(Remember that an i-factor of a graph (V, E) is a subgraph (V, E') where every vertex has degree i.)

The core of this approach is that the cost of the 2-factor of G_3 (for which we have a better approximation) cannot be dominated by the cost of the 1-factor of G_3. The main technical difficulty is in proving 1′.

4. A better approximation algorithm for the 3-ECSS/3-VCSS problem

In this section, we will present an approximation algorithm for the 3-ECSS/3-VCSS problem with an approximation ratio of $1 + \frac{5(2\beta-1)}{9(1-\beta)} + \mathcal{O}(\frac{1}{|V|})$ in case $\beta \leqslant \frac{2}{3}$.

We know from Lemma 1 that, for each two adjacent edges e, f of a graph obeying the sharpened triangle inequality with $\frac{1}{2} \leqslant \beta < 1$, $c(e) \leqslant \frac{\beta}{1-\beta} c(f)$, and vice versa. Since $\frac{\beta}{1-\beta} = 1 + \frac{2\beta-1}{1-\beta}$, we define
$$\gamma := \frac{2\beta - 1}{1 - \beta}$$
and obtain the reformulation
$$c(e) \leqslant (1+\gamma) c(f) \tag{2}$$

for each two adjacent edges e, f of a graph obeying the sharpened triangle inequality with $\frac{1}{2} \leqslant \beta < 1$.

First, we present an algorithm for the case of an even number of vertices, and later, we will extend this to the case of an odd number of vertices.

The idea of our algorithm is as follows. It first constructs a Hamiltonian cycle (using Algorithm Cycle Cover given in Section 2) and then connects each pair of opposite vertices of this cycle by an edge.

Algorithm 3C

Input: A complete weighted graph (G, c), with $G = (V, E)$, $|V| = n$ even, and $c : E \to \mathbb{Q}_{>0}$ obeying the sharpened triangle inequality for $\beta \leqslant \frac{2}{3}$.
Step 1: $H :=$ Hamiltonian cycle in G; (using Algorithm Cycle Cover)
Step 2: Let $v_1, v_2, \ldots, v_n$ be the order of vertices in H;
Step 3: $E_m := \{\{v_i, v_{i+\frac{n}{2}}\} \mid i = 1, \ldots, \frac{n}{2}\}$;
Output: $H \cup E_m$.

Algorithm 3C outputs a 3-vertex-connected subgraph, because it is just a special case of the output of Algorithm kC-Graph. We simply have the order of the vertices specially defined by H.

To show that Algorithm 3C has the approximation ratio stated above, we will make use of the following theorem (that we will prove in Section 5).

Theorem 5. *Let (G, c) be a complete weighted graph, where $G = (V, E)$, $|V| = n$ is even, and $c : E \to \mathbb{Q}_{>0}$ obeys the sharpened triangle inequality for $\beta \leqslant \frac{2}{3}$. Let $G' = (V, E')$ be a 3-edge-connected subgraph of G. Then, there exists a subgraph $G'' = (V, E'')$ of G that is 3-regular and 2-vertex-connected such that $\mathrm{cost}(G'') \leqslant \mathrm{cost}(G')$.*

Theorem 5 states that any optimal solution of the 3-ECSS/3-VCSS problem can be transformed into a 3-regular 2-vertex-connected(!) subgraph without increasing the cost. The main reason for doing this is that such a subgraph can be split into a 1-factor and a 2-factor.[2] By basically comparing these two parts separately to the corresponding parts of the constructed solution (the 1-factor vs E_m, the 2-factor vs H), we get the claimed result.

We need as assumptions an even vertex number and $\beta \leqslant \frac{2}{3}$ in order to apply Theorem 5. The case of odd vertex number is treated in Theorem 4.

Theorem 3. *Let (G, c) be an input for the 3-ECSS/3-VCSS problem, where $G = (V, E)$, $|V| = n$ is even, and $c : E \to \mathbb{Q}_{>0}$ obeys the sharpened triangle inequality for $\beta \leqslant \frac{2}{3}$. Then Algorithm 3C obtains an approximation ratio of $1 + \frac{5}{9}\gamma = 1 + \frac{5(2\beta-1)}{9(1-\beta)}$.*

Note that the approximation ratio is 1 for $\beta = \frac{1}{2}$, and it is $1 + \frac{5}{9}$ for $\beta = \frac{2}{3}$.

Proof. According to Theorem 5, because $\beta \leqslant \frac{2}{3}$ there exists a 3-regular and 2-vertex-connected subgraph $G'' = (V, E'')$ of G whose cost is less or equal to the cost of any 3-edge-connected subgraph of G, and thus is less or equal to the cost of any 3-vertex-connected subgraph of G.

By Petersen's Theorem (a consequence of Tutte's Theorem, see e.g. [15]), G'' contains a 1-factor (or perfect matching) M, and consequently the remainder of G'' is a 2-factor (or cycle cover) C. In the following, we obtain the claimed bound on the approximation ratio by essentially comparing E_m to M and H to C. But we have to distinguish two cases depending on whether C or M has the higher cost per edge.

Case 1. Let

$$\mathrm{cost}(C) \geqslant 2 \cdot \mathrm{cost}(M). \tag{3}$$

By (2), each edge $\{v, w\}$ of E_m costs at most $\frac{1}{2}(1+\gamma)(c(e_v) + c(e_w))$, where e_v and e_w are the edges of M incident to v and w, respectively. (Note that this includes the case $e_v = e_w = \{v, w\}$.) When we sum this up over all edges of E_m, each edge of M occurs exactly twice, which yields

$$\mathrm{cost}(E_m) \leqslant (1+\gamma)\mathrm{cost}(M). \tag{4}$$

Using the analysis of the approximation ratio of Algorithm Cycle Cover in [7], we have

$$\mathrm{cost}(H) \leqslant \left(1 + \frac{1}{3}\gamma\right)\mathrm{cost}(C), \tag{5}$$

[2] Note that for splitting off a 2-factor (or even a 1-factor) it is not sufficient that the considered graph is 3-edge-connected. A counterexample will be given at the end of the paper.

since exactly this inequality was established in [7] to obtain the approximation ratio of Algorithm Cycle Cover by using the cost of an optimal cycle cover as a lower bound on the cost of an optimal Hamiltonian cycle.

Put together, this gives the claimed bound

$$\begin{aligned}\mathrm{cost}(H)+\mathrm{cost}(E_m) &\underset{(4),(5)}{\leqslant} \left(1+\frac{1}{3}\gamma\right)\big(\mathrm{cost}(C)+\mathrm{cost}(M)\big)+\frac{2}{3}\gamma\cdot\mathrm{cost}(M)\\ &\underset{(3)}{\leqslant} \left(1+\frac{1}{3}\gamma\right)\big(\mathrm{cost}(C)+\mathrm{cost}(M)\big)+\frac{2}{3}\gamma\cdot\frac{1}{3}\big(\mathrm{cost}(C)+\mathrm{cost}(M)\big)\\ &= \left(1+\frac{5}{9}\gamma\right)\big(\mathrm{cost}(C)+\mathrm{cost}(M)\big).\end{aligned}$$

Case 2. Let

$$\mathrm{cost}(C)<2\cdot\mathrm{cost}(M). \tag{6}$$

Here, we compare each edge $\{v,w\}$ of E_m to all edges $e_{v,1}$, $e_{v,2}$, $e_{v,3}$, $e_{w,1}$, $e_{w,2}$, $e_{w,3}$ of $C\cup M$ incident to v and w, respectively. (Again, two of these may be identical to $\{v,w\}$.) By (2)

$$c\big(\{v,w\}\big)\leqslant\frac{1}{6}(1+\gamma)\left(\sum_{i=1}^{3}c(e_{v,i})+\sum_{i=1}^{3}c(e_{w,i})\right).$$

When we sum this up over all edges of E_m, each edge of $C\cup M$ occurs exactly twice, which yields

$$\mathrm{cost}(E_m)\leqslant\frac{1}{3}(1+\gamma)\big(\mathrm{cost}(C)+\mathrm{cost}(M)\big). \tag{7}$$

Using the approximation ratio of Algorithm Cycle Cover in the same way as described in (5), we have

$$\mathrm{cost}(H)\leqslant\left(1+\frac{1}{3}\gamma\right)\mathrm{cost}(C)\underset{(6)}{\leqslant}\left(1+\frac{1}{3}\gamma\right)\frac{2}{3}\big(\mathrm{cost}(C)+\mathrm{cost}(M)\big). \tag{8}$$

Put together, this gives as before the claimed bound

$$\begin{aligned}\mathrm{cost}(H)+\mathrm{cost}(E_m) &\underset{(7),(8)}{\leqslant} \left(\frac{1}{3}(1+\gamma)+\frac{2}{3}\left(1+\frac{1}{3}\gamma\right)\right)\big(\mathrm{cost}(C)+\mathrm{cost}(M)\big)\\ &= \left(1+\frac{5}{9}\gamma\right)\big(\mathrm{cost}(C)+\mathrm{cost}(M)\big). \qquad\Box\end{aligned}$$

Now, we describe how the previous result can be generalized to graphs with an odd number of vertices. In the following, let (G,c) be a complete weighted graph, where $G=(V,E)$, $V=\{v_1,\ldots,v_n\}$, n odd, and $c:E\to\mathbb{Q}_{>0}$ obeys the sharpened triangle inequality for $\beta\leqslant\frac{2}{3}$. Furthermore, let $c_{\min}$ and $c_{\max}$ denote the minimum and maximum edge cost in G, respectively. Let $(\tilde{G},c)$ be the complete weighted graph obtained by adding a new vertex x to G with edges of cost $\frac{\beta}{1-\beta}c_{\min}$ to all other vertices. More formally, $\tilde{G}=(\tilde{V},\tilde{E})$, where $\tilde{V}=V\cup\{x\}$ for some $x\notin V$ and $c(\{x,v\})=\frac{\beta}{1-\beta}c_{\min}$ for all $v\in V$.

The value of $\frac{\beta}{1-\beta}c_{\min}$, chosen for the new edges, is such that the β-triangle inequality holds even if edges of cost $c_{\min}$ and $c_{\max}$ both are adjacent to the new ones. This is shown in the proof of Lemma 1 in [7]. Consequently, $\tilde{G}$ as a whole obeys β-triangle inequality if G does.

The idea of the following algorithm is to use Algorithm 3C on $\tilde{G}$ and to transform the result to a 3-vertex-connected subgraph of G.

Algorithm 3C-Odd

Input: A complete weighted graph (G,c), where $G=(V,E)$, $|V|=n$ odd, and $c:E\to\mathbb{Q}_{>0}$ obeying the sharpened triangle inequality for $\beta\leqslant\frac{2}{3}$.

Step 1: Find a 3-vertex-connected subgraph $\tilde{A}$ of $\tilde{G}$ by Algorithm 3C.

Step 2: Obtain a 3-vertex-connected subgraph A of G by deleting the vertex x and its three incident edges from $\tilde{A}$, and inserting the three new edges between the three neighbors of x in $\tilde{A}$.

Output: A.

Theorem 4. *Let (G,c) be an input for the 3-ECSS/3-VCSS problem, where $G=(V,E)$, $|V|=n$ odd, and $c:E\to\mathbb{Q}_{>0}$ obeys the sharpened triangle inequality for $\beta\leqslant\frac{2}{3}$. Then, for all $n\geqslant 7$, Algorithm 3C-Odd obtains an approximation ratio of*

$$1+\frac{5}{9}\cdot\gamma+\frac{8\beta^2(\beta+4)}{9(1-\beta)^2}\cdot\frac{1}{n}=1+\frac{5(2\beta-1)}{9(1-\beta)}+\frac{8\beta^2(\beta+4)}{9(1-\beta)^2}\cdot\frac{1}{n}.$$

Proof. Since $\tilde{G}$ obeys the β-triangle inequality, the result of Theorem 3 is applicable for $\tilde{G}$. Moreover, it is obvious that the graph A constructed by Algorithm 3C-Odd is 3-edge-connected and 3-vertex-connected. Since Algorithm 3C-Odd deletes three edges of $\tilde{A}$ which are incident to x and adds three new edges, we can estimate the cost of A as follows:

$$\mathrm{cost}(A) \leqslant \mathrm{cost}(\tilde{A}) - 3\frac{\beta}{1-\beta}c_{\min} + 3c_{\max} \leqslant \mathrm{cost}(\tilde{A}) - 3\frac{\beta}{1-\beta}c_{\min} + 3\frac{2\beta^2}{1-\beta}c_{\min}. \tag{9}$$

The last inequality is from Lemma 1.

On the other hand, we can compare also the costs of the optimal solutions A_{opt} and $\tilde{A}_{\mathrm{opt}}$: Since adding the vertex x with three arbitrary incident edges to A_{opt} yields a feasible solution for $\tilde{G}$, we know that

$$\mathrm{cost}(A_{\mathrm{opt}}) \geqslant \mathrm{cost}(\tilde{A}_{\mathrm{opt}}) - 3\frac{\beta}{1-\beta}c_{\min}. \tag{10}$$

Using Eqs. (9) and (10), we can estimate the approximation ratio of Algorithm 3C-Odd as

$$\frac{\mathrm{cost}(A)}{\mathrm{cost}(A_{\mathrm{opt}})} \leqslant \frac{\mathrm{cost}(\tilde{A}) + 3\frac{2\beta^2-\beta}{1-\beta}c_{\min}}{\mathrm{cost}(\tilde{A}_{\mathrm{opt}}) - 3\frac{\beta}{1-\beta}c_{\min}}. \tag{11}$$

Let $\tilde{n} := n+1$. Since $\tilde{A}$ and $\tilde{A}_{\mathrm{opt}}$ each contain at least $\frac{3\tilde{n}}{2}$ edges, we have

$$c_{\min} \leqslant \frac{2}{3\tilde{n}}\mathrm{cost}(\tilde{A}_{\mathrm{opt}}) \quad \text{and} \quad c_{\min} \leqslant \frac{2}{3\tilde{n}}\mathrm{cost}(\tilde{A}). \tag{12}$$

Inserting Eq. (12) into Eq. (11) yields

$$\begin{aligned}
\frac{\mathrm{cost}(A)}{\mathrm{cost}(A_{\mathrm{opt}})} &\leqslant \frac{\mathrm{cost}(\tilde{A}) + 3\frac{2\beta^2-\beta}{1-\beta}\frac{2}{3\tilde{n}}\mathrm{cost}(\tilde{A})}{\mathrm{cost}(\tilde{A}_{\mathrm{opt}}) - 3\frac{\beta}{1-\beta}\frac{2}{3\tilde{n}}\mathrm{cost}(\tilde{A}_{\mathrm{opt}})} \\
&= \frac{\mathrm{cost}(\tilde{A})}{\mathrm{cost}(\tilde{A}_{\mathrm{opt}})} \cdot \frac{1 + \frac{2\beta^2-\beta}{1-\beta}\frac{2}{\tilde{n}}}{1 - \frac{\beta}{1-\beta}\frac{2}{\tilde{n}}} \\
&= \frac{\mathrm{cost}(\tilde{A})}{\mathrm{cost}(\tilde{A}_{\mathrm{opt}})} \cdot \left(1 + \frac{\frac{2\beta^2}{1-\beta}\frac{2}{\tilde{n}}}{1 - \frac{\beta}{1-\beta}\frac{2}{\tilde{n}}}\right) \\
&= \frac{\mathrm{cost}(\tilde{A})}{\mathrm{cost}(\tilde{A}_{\mathrm{opt}})} \cdot \left(1 + \frac{4\beta^2}{\tilde{n}(1-\beta) - 2\beta}\right) \\
&\leqslant \left(1 + \frac{5}{9}\cdot\gamma\right) \cdot \left(1 + \frac{4\beta^2}{\tilde{n}(1-\beta) - 2\beta}\right) \\
&= 1 + \frac{5}{9}\cdot\gamma + \frac{(1+\frac{5}{9}\gamma)4\beta^2}{\tilde{n}(1-\beta) - 2\beta},
\end{aligned}$$

where the second inequality follows from Theorem 3. To prove the desired estimation of the approximation ratio, we first prove the following claim for all $\frac{1}{2} \leqslant \beta \leqslant \frac{2}{3}$.

$$\tilde{n}(1-\beta) - 2\beta \geqslant \frac{1}{2}\tilde{n}(1-\beta), \quad \text{for all } \tilde{n} \geqslant 8. \tag{13}$$

For proving (13), it is sufficient to show $\tilde{n} \geqslant \frac{4\beta}{1-\beta}$, for all $\tilde{n} \geqslant 8$. We observe that the function $\frac{4\beta}{1-\beta}$ is monotone increasing between $\frac{1}{2}$ and $\frac{2}{3}$ and achieves its maximum value of 8 for $\beta = \frac{2}{3}$. This proves the claim (13). Recalling the definition of γ as $\gamma = \frac{2\beta-1}{1-\beta}$, this leads to

$$\begin{aligned}
\frac{\mathrm{cost}(A)}{\mathrm{cost}(A_{\mathrm{opt}})} &\leqslant 1 + \frac{5}{9}\cdot\gamma + \frac{(1+\frac{5}{9}\gamma)4\beta^2}{\frac{1}{2}\tilde{n}(1-\beta)} \\
&= 1 + \frac{5}{9}\cdot\gamma + \frac{8\beta^2(1+\frac{5}{9}\cdot\gamma)}{1-\beta}\cdot\frac{1}{\tilde{n}} \\
&= 1 + \frac{5}{9}\cdot\gamma + \frac{8\beta^2(1+\frac{5}{9}\cdot\frac{2\beta-1}{1-\beta})}{1-\beta}\cdot\frac{1}{\tilde{n}} \\
&= 1 + \frac{5}{9}\cdot\gamma + \frac{\frac{8}{9}\beta^2(\frac{9(1-\beta)+5(2\beta-1)}{1-\beta})}{1-\beta}\cdot\frac{1}{\tilde{n}}
\end{aligned}$$

$$= 1 + \frac{5}{9} \cdot \gamma + \frac{8\beta^2(\beta + 4)}{9(1-\beta)^2} \cdot \frac{1}{\tilde{n}},$$

for all $\tilde{n} > 8$. □

Note that for n tending to infinity the achieved approximation ratio tends to $1 + \frac{5}{9}\gamma$, i.e., to the approximation ratio in the case where n is even.

5. The proof of Theorem 5

Instrumental to the proof of the approximation ratio of Algorithm 3C is mainly the following theorem. It states that there exists a subgraph G'' that is not more expensive than any 3-edge-connected one, such that the structure of G'' is close to the solution constructed by our algorithm.

Here, we see why the assumption $\beta \leqslant \frac{2}{3}$ is needed. In view of Corollary 1, it allows to replace one edge by two adjacent ones without increasing the cost of the subgraph, which we will do frequently in the constructive proof.

Theorem 5. *Let (G, c) be a complete weighted graph, where $G = (V, E)$, $|V| = n$ is even, and $c : E \to \mathbb{Q}_{>0}$ obeys the sharpened triangle inequality for $\beta \leqslant \frac{2}{3}$. Let $G' = (V, E')$ be a 3-edge-connected subgraph of G. Then, there exists a subgraph $G'' = (V, E'')$ of G that is 3-regular and 2-vertex-connected such that $\operatorname{cost}(G'') \leqslant \operatorname{cost}(G')$.*

We will construct G'' from G' by successively deleting and adding edges in order to obtain degree exactly 3 for all vertices. This will be done in a way as to always maintain 2-edge-connectivity. In the end we have a 3-regular and 2-edge-connected graph. However, this is necessarily 2-vertex-connected, too, since any two edge-disjoint paths from u to w can never use a common vertex w in between. If they would, the distinct edges from both paths would give w degree 4 which is excluded in a 3-regular graph.

Maintaining 2-edge-connectivity will rely frequently on the following observations. The first one tells us which connections only need to be checked after modifying the subgraph.

Remark 1. Let G_1 be 2-edge-connected, and let G_2 result from deleting a few edges of G_1 (as well as potentially adding others). Then G_2 is 2-edge-connected iff for every pair of vertices v and w such that $\{v, w\}$ was deleted, there exist two edge-disjoint paths from v to w in G_2.

It is not trivial but easy to see that for any other pair of vertices x, y that needed the edge $\{v, w\}$ in one of its two connecting paths in G_1, two paths can be reestablished in G_2 by using the connections between v and w.

There is another easy observation which will be used frequently in the proof of Theorem 5. Here, we denote by $G \setminus v$ ($G \setminus u, v$) the rest of graph G after removal of v (u and v) together with the incident edges.

Remark 2. Let v be a vertex in a graph G' such that $G' \setminus v$ is 2-edge-connected. If there are two edges in G' between v and $G' \setminus v$, then G' is 2-edge-connected as well.

Finally we have the following statement about any connected graph.

Lemma 2. *If there are four distinct vertices $\{w_1, w_2, w_3, w_4\}$ in a connected graph G, there are two edge-disjoint paths in G having $\{w_1, w_2, w_3, w_4\}$ as endpoints.*

Proof. We just look at the tree of shortest paths from w_1 to w_2, w_3, and w_4. If the tree branches already at w_1, the path from w_1 to one of the other vertices and the remaining tree, including a path between the other two vertices, will have at most w_1 in common but no edge.

Otherwise, following the tree, starting from w_1, either we reach some $w_i, i \in \{2, 3, 4\}$ as an inner vertex, or we reach a branching where one of the branches leads to one vertex w_j out of $\{w_2, w_3, w_4\}$ only.

In the latter case the path $\langle w_1, \ldots, w_j \rangle$ and the remaining tree, including the path between the other two vertices, will have at most the branching vertex in common but no edge.

In the previous case, since w_i was reached before any branching or is the first branching point itself, the connection of the other two vertices may again have with the path $\langle w_1, \ldots, w_i \rangle$ at most the vertex w_i in common but no edge. □

Now, we are prepared for the main proof.

Proof of Theorem 5. As outlined above we want to construct G'' from G' by successively deleting and adding edges. This will be done in such a way that the cost is never increased, since whenever we introduce a new edge, it replaces two edges

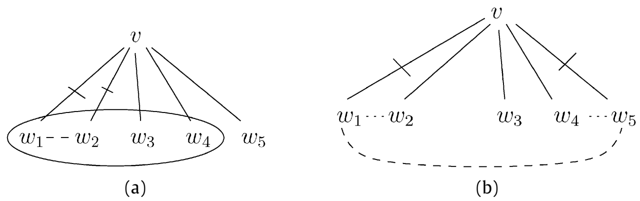

Fig. 2. Vertices of degree at least five: The ticked edges are removed, and the dashed edges are added. Dotted lines mark paths.

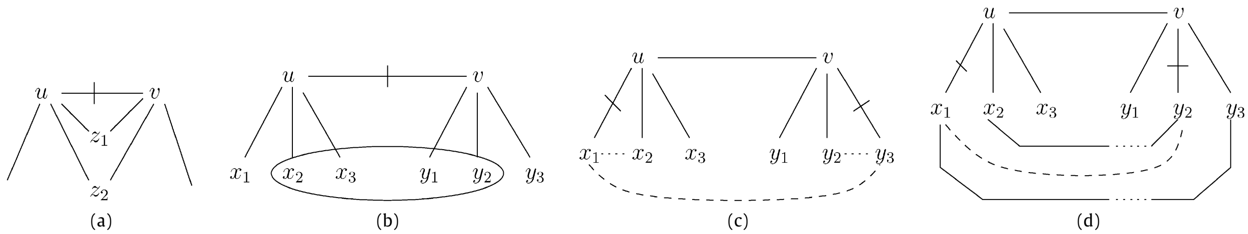

Fig. 3. Adjacent vertices of degree four.

adjacent to its endpoints. By Corollary 1, this does not increase the cost in case $\beta \leqslant \frac{2}{3}$, even if the deleted edges do not have their other endpoint in common.

Since G' is 3-edge-connected, it is also a 2-edge-connected graph where every vertex has degree at least 3. Thus, all we have to do is to reduce vertex degrees higher than 3 while maintaining the 2-edge-connectivity. The procedure obviously terminates since any modification decreases the number of remaining edges.

First, we deal with each vertex of degree 5 or more individually, and later, we will see how to handle vertices of degree 4, adjacent ones first. In the end, since we have an even vertex number, i.e. no single vertex of degree 4 may remain, all vertices will have degree 3.

A. *Vertices of degree* $\geqslant 5$. Let v be a vertex in a 2-edge-connected subgraph G_1 of G having at least five neighbors $w_1, \ldots, w_l, l \geqslant 5$ in G_1. Note that in the following, when speaking about connection or adjacency, we refer to G_1 unless stated otherwise.

We distinguish three cases.

A.1 Assume $G_1 \setminus v$ has a 2-edge-connected subgraph that contains at least four vertices out of $w_1, \ldots, w_l$, say w_1, w_2, w_3, w_4.
Then, we may delete two edges from v to these four vertices, without affecting the 2-edge-connectivity (cf. Remark 2). If w_1, w_2, w_3, w_4 form a clique, their degree still is at least 3. Otherwise, we choose two non-adjacent vertices, say w_1 and w_2. The edges $\{v, w_1\}$ and $\{v, w_2\}$ are deleted, and the edge $\{w_1, w_2\}$ is added to keep vertex degree at least 3 for w_1 and w_2 (see Fig. 2(a), where the ellipse stands for the 2-edge-connected subgraph).

A.2 Next, let $G_1 \setminus v$ be connected, but Sub-Case A.1 does not apply.
By Lemma 2, there are two edge-disjoint paths in $G_1 \setminus v$ between vertices from $\{w_1, \ldots, w_l\}$, w.l.o.g. $\langle w_1, \ldots, w_2\rangle$ and $\langle w_4, \ldots, w_5\rangle$.
If all edges between w_1, w_2 on one side, and w_4, w_5 on the other side would exist in G_1, those four vertices would be in a 2-edge-connected subgraph of $G_1 \setminus v$, a contradiction to the assumption that Sub-Case A.1 does not apply.
Let w.l.o.g. $\{w_1, w_5\}$ not exist in G_1. Now we replace $\{v, w_1\}$ and $\{v, w_5\}$ by $\{w_1, w_5\}$, obtaining G_2, see Fig. 2(b).
By symmetry and Remark 1, we only need to show that there are two edge-disjoint paths from w_1 to v in G_2 in order to prove that G_2 is 2-edge-connected. These paths are $\langle w_1, \ldots, w_2, v\rangle$ and $\langle w_1, w_5, \ldots, w_4, v\rangle$.

A.3 If $G_1 \setminus v$ is not connected, each vertex from $\{w_1, \ldots, w_l\}$ needs to be in the same component as at least one other from this set. The reason for this is the 2-edge-connectivity of G_1 which implies that from each component of $G_1 \setminus v$ there need to be two edges connecting that component with v.
So, we have again at least two connected pairs, and that the connecting paths are edge disjoint follows trivially from the fact that they are in different components. That fact also implies that the new to be introduced edge does not already exist in G_1. Thus, we may proceed exactly as in Sub-Case A.2.

B. *Adjacent vertices of degree* 4. Let u, v be two adjacent vertices of degree 4 in a 2-edge-connected subgraph G_1. We will distinguish four cases.

B.1 If u and v have at least two adjacent vertices in common, we simply can delete the edge $\{u, v\}$ (see Fig. 3(a)). Following Remark 1, the resulting graph is still 2-edge-connected.

In the following, let x_1, x_2, x_3 be adjacent to u, and let y_1, y_2, y_3 be adjacent to v. Only one x_i may be identical to one y_j.

B.2 Assume that there is a 2-edge-connected component in $G_1 \setminus u, v$ that contains at least two vertices each from the sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ respectively, four distinct vertices overall, say $\{x_2, x_3, y_1, y_2\}$. (Note that e.g. x_2 may be identical to y_3, but here it is used only in its role as x_2.) Again, we simply can delete $\{u, v\}$ (see Fig. 3(b), the ellipse represents the 2-edge-connected component). And again, following Remark 1, the resulting graph is still 2-edge-connected.

B.3 Next, let $G_1 \setminus u, v$ be not connected. By the 2-edge-connectivity of G_1, each of $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ must be in the same component with another vertex from that same set. Either there are at least two "mixed" pairs, say x_1, y_3 and x_2, y_2, or two pairs like x_1, x_2 and y_2, y_3. In the latter case, we delete $\{u, x_1\}$ and $\{v, y_3\}$ from G, and we insert $\{x_1, y_3\}$, obtaining G_2 (see Fig. 3(c)). By symmetry and Remark 1, we only need to show that there are two edge-disjoint paths from u to x_1 in order to prove G_2 to be 2-edge-connected. These paths are $\langle u, x_2, \dots, x_1\rangle$ and $\langle u, v, y_2, \dots, y_3, x_1\rangle$.
In case of mixed pairs, we replace $\{u, x_1\}$ and $\{v, y_2\}$ by $\{x_1, y_2\}$ (see Fig. 3(d)). Again, by symmetry and Remark 1, we only need to show that there are two edge-disjoint paths from u to x_1. These paths are $\langle u, x_2, \dots, y_2, x_1\rangle$ and $\langle u, v, y_3, \dots, x_1\rangle$.

B.4 If the previous sub-cases do not apply $G_1 \setminus u, v$ is connected. We can cut it such that two connected components remain, each containing at least two distinct vertices from $\{x_1, x_2, x_3, y_1, y_2, y_3\}$, such that the pairs are of the form either x_1, x_2 and y_2, y_3, or x_1, y_3 and x_2, y_2. (Note that we do not really delete edges but rather exclude them from use in paths. It is also important that all vertices have degree at most 4: this implies that the at least 5 distinct vertices from $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ cannot be connected in a star-like manner, and therefore we can make the cut as claimed.) Now, we can proceed exactly as in the previous sub-case except that we have to make sure that the introduced edge is not already existent.
But this is guaranteed by the fact that Sub-Case B.2 does not apply. First assume the pairs are x_1, x_2 and y_2, y_3. If every vertex from one pair would be adjacent to both from the other pair, the four of them would be in the same 2-edge-connected component. So, we can w.l.o.g. choose those non-adjacent vertices as being x_1 and y_3.
Now let the pairs be x_1, y_3 and x_2, y_2, and assume that both possible edges $\{x_1, y_2\}$ and $\{y_3, x_2\}$ would already exist in $G_1 \setminus u, v$. There are paths $\langle x_1, y_3\rangle$ and $\langle x_2, y_2\rangle$ that do not use $\{x_1, y_2\}$ and $\{y_3, x_2\}$, because x_1, y_3 and x_2, y_2 remain connected pairs after "cutting" the graph, which among other edges cuts $\{x_1, y_2\}$ and $\{y_3, x_2\}$.
Consequently $\langle x_1, y_3, x_2, y_2, x_1\rangle$ would be a simple circle in $G_1 \setminus u, v$. The four vertices would be in the same 2-edge-connected component, a contradiction. Hence, $\{x_1, y_2\}$ or $\{y_3, x_2\}$ does not exist in $G_1 \setminus u, v$. W.l.o.g. let that be $\{x_1, y_2\}$, and we can proceed as before.

C. *Independent vertices of degree* 4. Let u, v be two independent vertices in a 2-edge-connected subgraph G_1 having four neighbors each.

We have to distinguish similar cases as for adjacent vertices of degree 4 with some additional technical details.

Remember that we first dealt with all vertices of degree 5, and then with all adjacent vertices of degree 4. Consequently, we know that all the neighbors of u and v will have degree 3.

C.1 First assume that u and v have at least three adjacent vertices z_1, z_2, z_3 in common. Since z_1, z_2, z_3 have degree 3, at least one edge between them, say $\{z_1, z_2\}$, does not yet exist. Thus, we delete edges $\{u, z_1\}$ and $\{v, z_2\}$, and we insert $\{z_1, z_2\}$ (see Fig. 4(a)). Again, the result obviously is still 2-edge-connected and of degree at least 3.

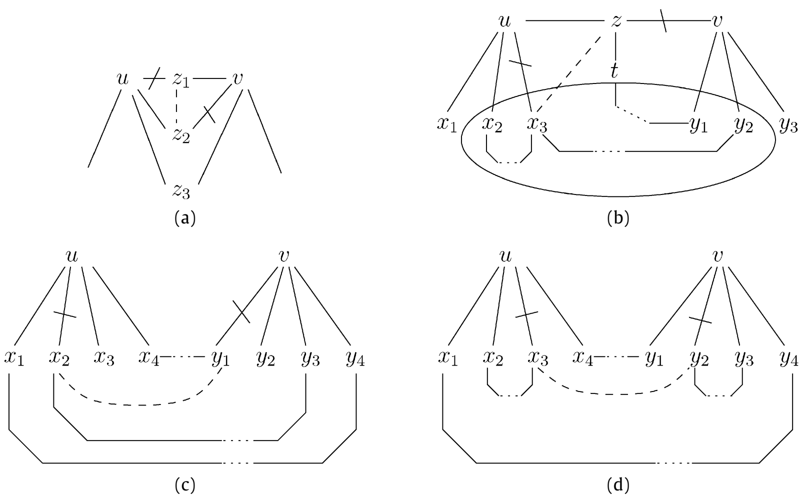

Fig. 4. Independent vertices of degree four.

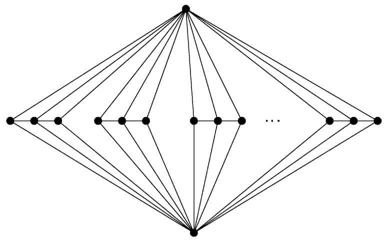

Fig. 5. 3-edge-connected graph without 1- or 2-factor.

In the following, let x_1, x_2, x_3, x_4 be adjacent to u, and let y_1, y_2, y_3, y_4 be adjacent to v. At most two of the x_i may be identical to some of the y_j.

C.2 Assume that there is at least one vertex $z = x_4 = y_4$ adjacent to both u and v.
Here, we refer to the sub-cases B.2–B.4 by using the two-edge path $\langle u, z, v\rangle$ in place of the single edge $\{u, v\}$ there. Note that it does not matter if z occurs on any path in $G_1 \setminus u, v$ considered there since we are focusing on edge-connectivity only. However, we need the following modifications.
In B.3, we use the argument that each of $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ must be connected to at least one other vertex from that set. Now, one of those at least five distinct vertices may be connected to z only (if two are connected to z, they are connected to each other). That leaves at least four vertices, hence the two needed pairs do exist.
In B.2, we delete the edge $\{u, v\}$ which is not possible here. Remember that we have four distinct vertices x_2, x_3, y_1, y_2, assumed to be in the same 2-edge-connected component of $G_1 \setminus u, v$, see the ellipse in Fig. 4(b).
Like all neighbors of u and v, z has three incident edges. Let the third one end in t. Since G was 2-edge-connected, there are paths from t to u and v that avoid z. We take one that contains just one of the vertices $x_1, x_2, x_3, y_1, y_2, y_3$: The first such vertex reached from t will do since it is connected to either u or v, and it might even happen that t itself is this vertex. W.l.o.g. let y_1 be that vertex. (It is obvious that the following consideration holds likewise if that vertex is one outside the 2-edge-connected component of $G_1 \setminus u, v$, say y_3.)
Since x_2, x_3, y_1, y_2 are in the same 2-edge-connected component of $G_1 \setminus u, v$ there is in that component a path $\langle x_3, \ldots, x_2\rangle$, as well as a path $\langle x_3, \ldots, y_2\rangle$ that has no edge in common with the path $\langle t, \ldots, y_1\rangle$ mentioned above.
Now we replace the edges $\{z, v\}$ and $\{u, x_3\}$ by $\{z, x_3\}$, see Fig. 4(b). The two edge-disjoint paths between x_3 and u are $\langle x_3, \ldots, x_2, u\rangle$ and $\langle x_3, z, u\rangle$. And from z to v we have $\langle z, x_3, \ldots, y_2, v\rangle$ and $\langle z, t, \ldots, y_1, v\rangle$.

C.3 It remains the case that $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$ are disjoint sets.
First, the 2-edge-connectivity of G_1 guarantees that there are at least two edge-disjoint paths from some x_i to some y_j, say from x_1 to y_4 and from x_4 to y_1.
Next, assume that there is even a third edge-disjoint path, say from x_2 to y_3. Since all neighbors of u and v have degree 3, one of the edges between a vertex from $\{x_1, x_2, x_4\}$ and one from $\{y_1, y_3, y_4\}$ does not exist in G_1, say $\{x_2, y_1\}$. We replace $\{u, x_2\}$ and $\{v, y_1\}$ by $\{x_2, y_1\}$, obtaining G_2 (see Fig. 4(c)). By symmetry and Remark 1, we only need to show that there are two edge-disjoint paths from u to x_2 in order to prove G_2 to be 2-edge-connected. These paths are $\langle u, x_4, \ldots, y_1, x_2\rangle$ and $\langle u, x_1, \ldots, y_4, v, y_3, \ldots, x_2\rangle$.
Finally, if no third edge-disjoint path for some x_i to some y_j exists, the 2-edge connectivity of G_1 guarantees that x_2, x_3 each are connected to at least one other vertex from $\{x_1, x_2, x_3, x_4\}$, and y_2, y_3 each are connected to at least one other vertex from $\{y_1, y_2, y_3, y_4\}$.
If both x_2 and x_3 would be connected to x_1 or would use edges from the path $\langle x_1, \ldots, y_4\rangle$ in their connections, we could rearrange the situation (potentially by exchanging x_1 with x_2 or x_3) such that we obtain, after renaming, edge-disjoint paths $\langle x_1, \ldots, y_4\rangle$ and $\langle x_2, \ldots, x_3\rangle$. Otherwise, either x_2 and x_3 are connected to each other or one of them to x_4 by a path disjoint from $\langle x_1, \ldots, y_4\rangle$. (In the latter case, we rename the vertices x_2, x_3, x_4 such that the path in question is $\langle x_2, \ldots, x_3\rangle$.) A similar consideration applied to $\{y_1, \ldots, y_4\}$ yields, potentially after renaming, three edge-disjoint paths $\langle x_1, \ldots, y_4\rangle$, $\langle x_2, \ldots, x_3\rangle$ and $\langle y_2, \ldots, y_3\rangle$.
Now, we replace $\{u, x_3\}$ and $\{v, y_2\}$ by $\{x_3, y_2\}$ (see Fig. 4(d)). Since there was no third path, the edge $\{x_3, y_2\}$ did not exist before.[3] Also, the new subgraph is 2-edge-connected. To show this, the existence of two edge-disjoint paths $\langle u, x_2, \ldots, x_3\rangle$ and $\langle u, x_1, \ldots, y_4, v, y_3, \ldots, y_2, x_3\rangle$ is sufficient by symmetry and Remark 1. □

Finally, we present an example showing that in general one cannot split off a 2-factor (or even a 1-factor) from a 3-edge- or 3-vertex-connected graph. If this was the case, then our construction of a 3-regular 2-vertex-connected graph from an optimal 3-edge-connected one, as given in the proof of Theorem 5, would not be necessary. Thus, the graph in Fig. 5 shows that the construction of Theorem 5 is essential. The presented graph is 3-edge-connected, but it is impossible to split off a 2-factor or even a 1-factor.

[3] This is not affected by the renaming, since if the path $x_1, \ldots, y_4$ consisted of a single edge, we would not have needed to rename those vertices.

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